A matrix model for random surfaces with dynamical holes

1996 J. Phys. A: Math. Gen. 29 3769

(http://iopscience.iop.org/0305-4470/29/14/006)

J. Phys. A: Math. Gen. **29** (1996) 3769–3785. Printed in the UK

A matrix model for random surfaces with dynamical holes

Giovanni M Cicuta†‖, Luca Molinari‡¶, Emilio Montaldi‡ and Sebastiano Stramaglia§+

† Dipartimento di Fisica, Università di Parma and INFN, Sezione di Milano, Viale delle Scienze, 43100 Parma, Italy

‡ Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy

§ Dipartimento di Fisica, Università di Bari and INFN, Sezione di Bari, Via Amendola 173, 70126 Bari, Italy

Received 5 June 1995, in final form 24 January 1996

Abstract. A matrix model to describe dynamical loops on random planar graphs is analysed. It has similarities with a model studied by Kazakov, a few years ago, and the $O(n)$ model of Kostov and collaborators. The main difference is that all loops are coherently oriented and empty. The free energy is analytically evaluated and the continuum limit is analysed in a region of parameters where the universality of the continuum description may not be expected. Our phase diagram is analogous to Kazakov's model with two phases (surface with small holes and tearing phase) with Kazakov's scaling exponents. The critical exponents of the third phase, which occurs on the boundary between the two above phases, differ from the corresponding exponents in Kazakov's model [1].

1. Introduction

Field theory models with matrix variables have been the focus of a very large number of investigations in the past decade. The analysis of these models in the limit of large order of the matrices, the large-N limit, even in reduced dimension of spacetime, provides important suggestions for the non-perturbative understanding of quantum field theory and for the formulation of string theory.

At the beginning of the recent developments, much attention was given to the loop correlators

$$W(l_1, l_2, \ldots, l_m) = \frac{1}{N^m}\langle(\mathrm{Tr}\,\mathrm{e}^{l_1\Phi})\ldots(\mathrm{Tr}\,\mathrm{e}^{l_m\Phi})\rangle_V = \frac{1}{Z}\int \mathcal{D}\Phi\,\mathrm{e}^{-N\,\mathrm{Tr}\,V(\Phi)}\frac{1}{N^m}\,\mathrm{Tr}(\mathrm{e}^{l_1\Phi})\ldots\mathrm{Tr}(\mathrm{e}^{l_m\Phi}) \tag{1.1}$$

and their Schwinger–Dyson equations. In the simplest case the potential $V(\Phi)$ is a polynomial in the Hermitian matrix variable Φ. The Laplace transform of the above correlators

$$\tilde{W}(p_1, p_2, \ldots, p_m) = \frac{1}{N^m}\left\langle \mathrm{Tr}\left(\frac{1}{p_1-\Phi}\right)\ldots\mathrm{Tr}\left(\frac{1}{p_m-\Phi}\right)\right\rangle_V \tag{1.2}$$

‖ CICUTA@PARMA.INFN.IT
¶ MOLINARI@MILANO.INFN.IT
+ STRAMAGLIA@BARI.INFN.IT

0305-4470/96/143769+17$19.50

satisfy a chain of equations where multi-loop correlators may be evaluated from correlators with fewer loops, and correlators of higher genus in the topological expansion are evaluated from correlators of lower genus [2–5].

The Schwinger–Dyson equation for the one-loop correlator

$$N V'\left(\frac{\partial}{\partial l}\right) W(l) = \int_0^l \mathrm{d}l'\, W(l', l-l') \tag{1.3}$$

implies, in the planar limit, a quadratic equation for its Laplace transform, with a solution equivalent to the more usual saddle-point analysis.

Loop correlators correspond to insertions of loops of perimeters $l_1, \ldots, l_m$ in the dynamical triangulation provided by the graphs dual to the planar Feynman graphs of the model. Loop equations have a pictorial interpretation as splitting or gluing of loops and handles. In the continuum limit of the matrix model, obtained for critical values of the couplings in the potential, loops may be arranged to have finite or infinitesimal lengths, thus being referred to as macroscopic or, respectively, microscopic loops.

A related analysis was done for the operator $\psi_n = \mathrm{Tr}(\Phi^n)$, corresponding to the insertion of an n-sided polygon in the dynamical triangulation. Duplantier and Kostov analysed the connected two-point correlator $\langle \mathrm{Tr}(\Phi^L)\,\mathrm{Tr}(\Phi^L)\rangle_V$, and obtained critical coefficients for the problem of random self-avoiding paths on random planar graphs [6].

In a very interesting paper, Kazakov [7] analysed the effects of *dynamical* loops in the simplest one-matrix model, in the planar limit. The partition function is

$$Z = \int \mathcal{D}M \exp\{-N\,\mathrm{Tr}[\tfrac{1}{2}M^2 - \tfrac{1}{4}gM^4 + L\log(1 - z^2M^2)]\}. \tag{1.4}$$

(We slightly change Kazakov's notation to simplify the comparison with the present paper.) In the large-N limit, the free energy $E = -\frac{1}{N^2}\log Z$ is the sum of planar Feynman graphs where 'gluons' interact with the quartic vertex g and, in the proper continuum limit, describe planar connected surfaces with the insertion of an arbitrary number of holes of arbitrary lengths. The parameter L may be regarded as the hole fugacity. In the large-N limit, by a saddle-point analysis, he showed that the model has three different phases in the continuum limit, characterized by different scaling behaviours as $L \to 0$ of the average number of holes $\langle h\rangle$ and of the total perimeter of the holes $\langle l\rangle$, both observables being evaluated per unit area:

(i) The 'small-holes phase', which occurs for $g/z^2 > \frac{2}{3}$, where holes are rare ($\langle h\rangle \to 0$) and the average length of one hole $\langle l\rangle/\langle h\rangle$ approaches a finite value while the surface area diverges.

(ii) The 'tearing phase', which occurs for $g/z^2 < \frac{2}{3}$, where the surface is almost filled with large and dense holes. In the limit $L \to 0$, $\langle l\rangle$ approaches a non-vanishing constant, quite like an order parameter of a spontaneous symmetry breaking. This finite value, due to the diverging perimeter of the average hole while the number of holes vanishes ($\langle h\rangle \sim L^{2/3}$), may be called the residual total perimeter.

(iii) The border line in the parameter space, separating the two above-mentioned two-dimensional manifolds, at $g/z^2 = \frac{2}{3}$, provides another scaling behaviour $\langle h\rangle \sim L^{4/5}$, $\langle l\rangle \sim L^{2/5}$. The average length diverges as in the 'tearing phase', but the residual total perimeter $\langle l\rangle$ vanishes as in the 'small-holes phase'.

The same model was later analysed by Kostov [8] and Minahan [9], with the technique of orthogonal polynomials. The role of fermions in generating dynamical loops in matrix models of Kazakov type, equation (1.4), was investigated in one dimension [10, 11]. A similar analysis was also done in unitary matrix models with boundary terms [9, 12].

Field theory models of surfaces with dynamical loops are interesting for the formulation of field theories of interacting strings. They are usually called field theories of open strings, but also have implications for models of surfaces with handles and no boundaries, usually referred to as field theories of closed strings. Indeed, after a proper identification of h pairs of boundaries, the partition function associated to a surface with $2h$ boundaries must be equal to the partition function associated with a surface with h handles. In the past few years, large-N QCD on a generic two-dimensional manifold has been investigated as a string model and as a topological theory [13–16] and some identities of this type were exhibited.

This paper is an investigation of a closely related random matrix model which seems promising for the description of two-dimensional manifolds with oriented boundaries. The present model is the most straightforward analogue, in zero dimension of spacetime, of the Veneziano multi-flavour chromodynamics, with the gluon field replaced by a Hermitian $N \times N$ matrix M and the L-flavoured fermions replaced by a set of L complex $N \times N$ matrices ϕ_a, $a = 1, \ldots, L$.

The partition function of our model is

$$Z_N(L, z) = \int \mathcal{D}M \prod_{a=1}^{L} \mathcal{D}\phi_a \exp\left\{ -N\,\mathrm{Tr}\left[V(M) + \frac{1}{2}\sum_{a=1}^{L} \phi_a^\dagger \phi_a - z\sum_{a=1}^{L} (M\phi_a^\dagger \phi_a) \right]\right\} \qquad (1.5a)$$

with $z > 0$ and the usual integration measures for Hermitian and complex matrices

$$\mathcal{D}M = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}(\mathrm{Re}\, M_{ij})\, \mathrm{d}(\mathrm{Im}\, M_{ij}) \qquad \mathcal{D}\phi = \prod_{i,j=1}^{N} \mathrm{d}(\mathrm{Re}\, \phi_{ij})\, \mathrm{d}(\mathrm{Im}\, \phi_{ij}).$$

Our choice of potential is

$$V(M) = \tfrac{1}{2}M^2 + \tfrac{1}{3}gM^3 \qquad (1.5b)$$

By performing the Gaussian integration over the complex matrices ϕ_a, and neglecting an irrelevant constant, the partition function (1.5a) is rewritten as

$$Z_N(L, z) = \int \mathcal{D}M \exp\{-N\,\mathrm{Tr}[V(M) + L\log(1 - 2zM)]\}. \qquad (1.5c)$$

As is apparent from (1.5), the model, in the large-N limit, describes connected planar surfaces, generated by 'gluons' with cubic interactions, with an arbitrary number of coherently oriented non-intersecting closed boundaries (holes), generated by the propagators of the charged fields ϕ_a (figure 1).

The signs of the various couplings in the action deserve a comment. Both signs of L are interesting: the positive sign in (1.4) and (1.5c) is usually referred to as the bosonic case, the opposite sign would correspond to boundaries originated by fermionic fields. In the model (1.5b), (1.5c), as is seen by changing M into $-M$, there are only two inequivalent cases according to $gz > 0$ or $gz < 0$. In the latter case, let us consider $g < 0$ and $z > 0$. It follows that the coefficients c_k in the expansion of the action $A = \sum c_k M^k$ are all negative for $k > 2$. This situation also occurs in Kazakov's model (1.4) and one expects critical coefficients in the same class of universality.

The present paper is concerned with the case $g > 0$, $z > 0$. Because of the non-constant sign of the coefficients c_k, one cannot predict critical coefficients to be in the same universality class. In a way quite analogous to Kazakov's model, three phases are found. The scaling behaviour of relevant observables is the same for phases (i) and (ii), but it is different for phase (iii). This non-universal behaviour is the main result of the present paper.

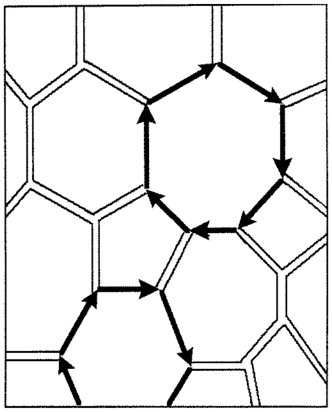

Figure 1. A portion of a planar graph generated by the model (1.5*a*). The loops formed by black arrows are generated by the propagators of the charged fields.

The L-expansion of the free energy

$$E(L,z) = -\lim_{N\to\infty} \frac{1}{N^2} \log Z_N = E_0 + \sum_{k=1}^{\infty} L^k E_k(z) \tag{1.6}$$

provides the generators $E_k(z)$ of connected planar graphs with k holes. The further expansion of E_k in powers of z classifies graphs according to the total perimeter of the k holes, measured as the number of connections with the surrounding surface. The term E_0 is the planar free energy of the cubic one-matrix model, solved some time ago [17].

In the Boltzmann factor in (1.5*a*) one could add the term providing the other orientation,

$$\frac{z}{\sqrt{N}} \sum_{a=1}^{L} \mathrm{Tr}(\phi_a^\dagger M \phi_a)$$

to obtain

$$\mathrm{Tr} \sum_{1}^{L} \left\{ \frac{1}{2}\phi_a^\dagger \phi_a - \frac{z}{\sqrt{N}} M(\phi_a^\dagger \phi_a + \phi_a \phi_a^\dagger) \right\} = \mathrm{Tr} \sum_{1}^{2L} \left[\frac{1}{2}\Phi_a \Phi_a - \frac{2z}{\sqrt{N}} M \Phi_a \Phi_a \right] \tag{1.7}$$

where Φ_a is the set of $2L$ Hermitian $N \times N$ matrices defined by the Hermitian and the anti-Hermitian components of ϕ_a. One would then obtain the partition function of the $O(2L)$ vector model on a random lattice [6, 18–20]. In the large-N limit, its free energy describes connected surfaces with any number of non-oriented, self-avoiding loops which *are not holes*.

The paper is organized as follows. In section 2 we compute the free energy for the cubic interaction with charged loops and introduce the thermodynamic quantities. Next, in section 3, we explore its critical behaviour. We also give a simple theorem to show the connection of the edge behaviour of the eigenvalue density with the critical behaviour of the parameters for its support. In section 4, we provide an independent analysis with orthogonal polynomials. The non-universality found for the boundary phase (iii), related to the non-equivalence of the critical points with opposite signs of the cubic coupling, is explained. In section 5 we show the relationship of the one-hole term, with generic potential, with the leading asymptotics of orthogonal polynomials. Section 6 summarizes the conclusions.

2. The free energy for connected random surfaces with holes

In this section we proceed to evaluate the large-N limit of the free energy of the model in (1.5). According to the standard procedure, to study the large-N behaviour of Z_N in (1.5*c*), one performs the change of variables from the set M_{ij} to eigenvalues λ_i and angles, which can be integrated. The partition function, with irrelevant constants removed, is

$$Z_N = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left\{ -N \sum_{i=1}^{N} [V(\lambda_i) + L \log(1 - 2z\lambda_i)] \right\}. \tag{2.1}$$

The integral is dominated by a saddle-point configuration, described by a normalized density $\rho(\lambda)$ with support (a, b), that solves the singular integral equation

$$\lambda + g\lambda^2 - \frac{2zL}{1 - 2z\lambda} - 2P \int_a^b \mathrm{d}\mu \, \frac{\rho(\mu)}{\lambda - \mu} = 0. \tag{2.2}$$

The solution is

$$\rho(\lambda) = \frac{1}{2\pi} \sqrt{(b-\lambda)(\lambda - a)} \left[g\lambda + (gs+1) - z \frac{2s(1+gs) + gd^2}{1 - 2z\lambda} \right] \tag{2.3}$$

where $s \equiv (a+b)/2$ and $d \equiv (b-a)/2$ are determined by the two equations

$$L - 2 = s(1+gs)\left(\frac{1}{2z} - s\right) - \frac{d^2}{2}\left(1 + 3gs - \frac{g}{2z}\right) \tag{2.4a}$$

$$2zL = \left(s + gs^2 + \frac{g}{2}d^2\right)\sqrt{(1-2zs)^2 - 4z^2d^2}. \tag{2.4b}$$

Of course, the solution (2.3) holds if the pole $\lambda = 1/2z$ is outside the support. Since we choose $z > 0$, we require $1/2z > b$. After a long computation we obtain a simplified, yet not inspiring, expression for the free energy

$$\begin{aligned} E(g, L, z) = {} & -\log\left(\frac{d}{2}\right) + L \log\left[\frac{d}{2}(h + \sqrt{h^2 - 1})\right] - \frac{L^2}{2} \log\left[\frac{h + \sqrt{h^2-1}}{\sqrt{h^2 - 1}}\right] \\ & + L\frac{d^2}{16}(1+2gs)\left[2 - \left(h - \sqrt{h^2-1}\right)^2\right] + \frac{Lg}{4}d^3\left[\frac{h^3}{3} - \frac{h}{2} - \frac{(h^2-1)^{3/2}}{3}\right] \\ & - \frac{L}{2} + \frac{1}{16z^2}\left(\frac{g}{3z} + 1\right)\left[1 - \frac{d^2}{4}(1+2gs)\right] + \frac{g^2d^6}{192} + \frac{d^4}{64}[1 + 6gs + 6g^2s^2] \\ & + \frac{d^2}{4}\left[s(1+gs) + \frac{g}{2}d^2\right]\left[\frac{gd^2}{24} + \left(\frac{gs}{6} + \frac{1}{4}\right)\left(\frac{1}{2z} + s\right) + \frac{g}{24z^2}\right] \\ & + \frac{d^2}{48}(1+2gs)(2gs^3 + 3s^2 + 6) + \frac{s^2}{4} + \frac{gs^3}{6} \qquad h \equiv \frac{1}{d}\left(\frac{1}{2z} - s\right). \end{aligned} \tag{2.5}$$

For later use, and as a check of the above expression, we computed the first terms of the Taylor expansion in L of the free energy. The details are given in the appendix.

It is convenient to introduce the fugacity of the holes, perimeter $\tau = g/(2z)$. In the expansion

$$E(g, L, \tau) = \sum_{k=1}^{\infty} g^k E_k(L, \tau) \tag{2.6}$$

where $E_k(L, \tau)$ corresponds to the partition function of a statistical system of loops on random planar lattices with k sites. $g_{\mathrm{cr}}(L, \tau)$ being the radius of convergence of the series, in the thermodynamic limit the free energy per site is

$$f(L, \tau) = \lim_{k \to \infty} \frac{1}{k} \log E_k(L, \tau) = \log g_{\mathrm{cr}}(L, \tau) \tag{2.7}$$

which allows us to evaluate the average number of holes per site

$$\langle h \rangle = \frac{\partial \log g_{\rm cr}(L,\tau)}{\partial \log L} \tag{2.8}$$

and the average length of the total perimeter

$$\langle l \rangle = -\frac{\partial \log g_{\rm cr}(L,\tau)}{\partial \log \tau}. \tag{2.9}$$

Before closing this section we briefly recall the simpler model equations (1.5) with $g = 0$. A few terms of the L expansion of the planar free energy were evaluated long ago [21] and provide a non-trivial check for the more involved algebra of the present paper.

When $g = 0$ one may perform the Gaussian integration of the matrix M in equation (1.5) to obtain, neglecting irrelevant constants, the partition function for a set of L complex matrices ϕ_a with quartic coupling

$$Z_N(L,z) = \int \prod_{a=1}^{L} \mathcal{D}\phi_a \exp\left\{ -\,\mathrm{Tr}\left[\frac{1}{2}\sum_{a=1}^{L}\phi_a^\dagger\phi_a - \frac{z^2}{2N}\left(\sum_{a=1}^{L}\phi_a^\dagger\phi_a\right)^2 \right]\right\}. \tag{2.10}$$

There are advantages in regarding $Z_N(L,z)$ as a model of just one rectangular matrix Φ, of dimension $NL \times N$ with random complex entries. In this way the model was solved both in the planar limit [22] and, by the orthogonal polynomial technique, in the $1/N$ expansion [23]. This approach clarifies an interesting symmetry of Green functions under the exchange $L \to 1/L$, which may also have a bearing in the present case. For square matrices, that is $L = 1$, it is well known [17] that the model should be analysed in the large-N limit only for $0 \leqslant z^2 \leqslant z_{\rm cr}^2 = \frac{1}{48}$. For generic L, the bound $z^2 \leqslant z_{\rm cr}^2(L)$ is the special case $g = 0$ of (3.9). In figure 2 we plot the critical line, and exhibit the point $L = 1$, $1/(2z) = 2\sqrt{3}$. The finite arc with $0 < L < 1$ is mapped into the infinite arc $L \geqslant 1$ through the symmetry $(L,\xi) \Leftrightarrow (1/L, \xi/\sqrt{L})$, where $\xi = 1/2z$.

We remark that the saddle point (2.2), after the shift $x = \lambda - s$, is easily rewritten as a system of two equations for the even and odd components of the eigenvalue density

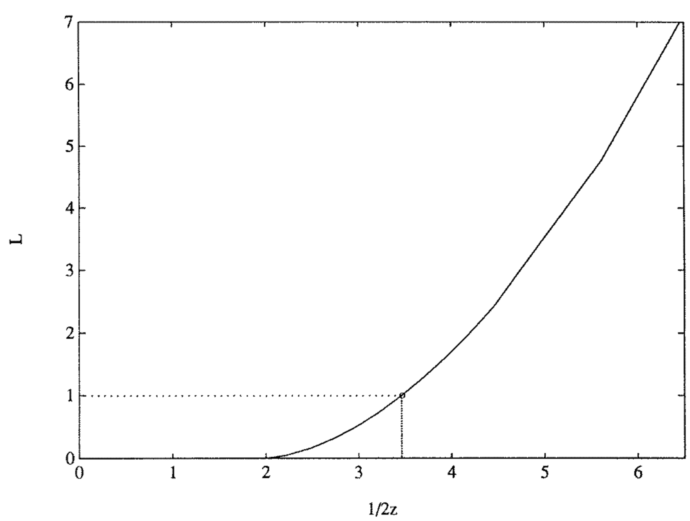

Figure 2. The critical line $z^2 = z_{\rm cr}^2(L)$ of the model (2.10).

$\rho(x) = \rho_e(x) + \rho_o(x)$, where $\rho_e(x) = \rho_e(-x)$ and $\rho_o(x) = -\rho_o(-x)$:

$$gx^2 + s(1+gs) - \left(\frac{1}{2z} - s\right)\frac{L}{(1/2z-s)^2 - x^2} = 2P\int_{-d}^{d} \mathrm{d}y\,\frac{\rho_o(y)}{x-y} \tag{2.11a}$$

$$x(1+2gs) - \frac{Lx}{(1/2z-s)^2 - x^2} = 2P\int_{-d}^{d} \mathrm{d}y\,\frac{\rho_e(y)}{x-y}. \tag{2.11b}$$

As long as $0 \leqslant z^2 \leqslant z_{\mathrm{cr}}^2(L)$ the coupled equations (2.11) are trivially equivalent to (2.2). One may, however, consider the analytic continuation for $z^2 \leqslant 0$, which in the simpler model (2.10) corresponds to the usually 'correct' real positive value for the quartic coupling. Since z would be pure imaginary, the saddle-point equation (2.2) becomes complex and (2.11) suggest the proper path in the complex plane, as was evaluated in [21].

3. Phases and continuum limits

In this section we describe the singularities of the free energy with respect to the couplings, which lead to different phases for the model and distinct continuum limits.

We recall that in one-matrix models where the potential is a polynomial in the matrix variable, one generally finds that such singularities are determined by the condition that the variables specifying the support are singular functions of the couplings in the potential. This in turn is equivalent to the requirement that the eigenvalue density vanishes at the end of the support with a zero of order $n + \frac{1}{2}$, with the integer n larger than zero.

These properties of one-matrix models are well known and were proved using orthogonal polynomials by Itzykson and Zuber [24]. However, phase transitions are more conveniently discussed in the saddle-point approach, and the simple proof provided here also sheds light on the limitations of the assertions.

Let us consider a polynomial potential $V(\lambda) = \sum_k g_k \lambda^k$. The saddle-point equation for the normalized density $\rho(\lambda)$ can be solved by the Poincaré–Bertrand inversion formula [25]. In the phase where the support is a single segment (a, b), the solution may be written as

$$\begin{aligned}\rho(\lambda) &= \frac{1}{\pi}\frac{1}{\sqrt{(b-\lambda)(\lambda-a)}}\mathcal{P}(\lambda, b, a, g_i)\\ \mathcal{P}(\lambda, b, a, g_i) &= 1 + \frac{1}{2\pi}P\int_a^b \mathrm{d}\mu\,\sqrt{(b-\mu)(\mu-a)}\,\frac{V'(\mu)}{\mu-\lambda}\end{aligned} \tag{3.1}$$

where $\mathcal{P}(\lambda, b, a, g_i)$ is a polynomial in the variable λ whose coefficients are entire functions of b, a and g_i. The end points $a(g_i), b(g_i)$ of the support are determined by the conditions

$$\mathcal{P}(\lambda = a, b, a, g_i) = \mathcal{P}(\lambda = b, b, a, g_i) = 0. \tag{3.2}$$

Inserting the identity

$$\frac{\sqrt{(b-\mu)(\mu-a)}}{\mu-\lambda} = \frac{(b-\lambda)(\lambda-a)}{(\mu-\lambda)\sqrt{(b-\mu)(\mu-a)}} + \frac{a+b-\lambda-\mu}{\sqrt{(b-\mu)(\mu-a)}}$$

in equation (3.1) and imposing the conditions (3.2), one obtains the following equations for the extrema:

$$\frac{1}{2\pi}\int_a^b \mathrm{d}\mu\,\frac{V'(\mu)}{\sqrt{(b-\mu)(\mu-a)}} = 0 \qquad \frac{1}{2\pi}\int_a^b \mathrm{d}\mu\,\frac{\mu V'(\mu)}{\sqrt{(b-\mu)(\mu-a)}} = 1 \tag{3.3}$$

and the factorization

$$\mathcal{P}(\lambda, b, a, g_i) = (b-\lambda)(\lambda-a)\mathcal{Q}(\lambda)$$
$$\mathcal{Q}(\lambda) = \frac{1}{2\pi}\int_a^b \mathrm{d}\mu\, \frac{V'(\mu)-V'(\lambda)}{\mu-\lambda}\frac{1}{\sqrt{(b-\mu)(\mu-a)}}. \tag{3.4}$$

The above formula implies that, generically, the density $\rho(\lambda)$ vanishes with a zero of order one half at the extrema of its support.

Since the free energy $E(g_i)$ may be evaluated as a polynomial functional of $\rho(\lambda)$ and because of the polynomial nature of $\mathcal{Q}(\lambda, b, a, g_i)$, the singularities of $E(g_i)$ may only occur as singularities of the functions $a(g_i)$ and $b(g_i)$. By differentiating equations (3.2) with respect to any free parameter g_i, and carrying the computations of the required partial derivatives in the parameters a, b in (3.1) and in the variable λ on formula (3.4), one eventually finds

$$\frac{\partial a}{\partial g_i} = -2\frac{(\partial\mathcal{P}/\partial g_i)_{\lambda=a}}{(b-a)\mathcal{Q}(a)} \qquad \frac{\partial b}{\partial g_i} = 2\frac{(\partial\mathcal{P}/\partial g_i)_{\lambda=b}}{(b-a)\mathcal{Q}(b)}. \tag{3.5}$$

Therefore, the singularities of the left-hand sides of equations (3.5) may only occur at the zeros of the denominators of the right-hand sides. These, in turn, imply a non-generic order for the vanishing of the density at the edge of its support.

We recall important examples where the above argument is evaded. In the attempt to describe random surfaces with extrinsic curvature, matrix models were proposed where the potential $V(M)$ of the Hermitian matrix M is a sum of monomials and the same invariant trace occurs with different powers [26]. The simplest example is

$$V(M) = a\,\mathrm{Tr}(M^2) + b\,\mathrm{Tr}(M^4) + c[\mathrm{Tr}(M^2)]^2. \tag{3.6}$$

The eigenvalue density is easily found in the large-N limit by the saddle-point analysis. However, more parameters determined by more equations occur, and the general features described above in equations (3.1)–(3.4) must be generalized. Indeed it was shown that these models yield susceptibilities with unusual critical coefficients [26, 27].

A second class of models which escape the above theorem occur if the one-matrix potential is not a polynomial. This is the case for the Kazakov model and for the present paper. Then the function $\mathcal{Q}(\lambda, b, a, g_i)$ in (3.4) is not a polynomial and the singularities in (3.5) may arise from singularities of the numerator in the right-hand side of the equation.

We proceed to analyse the critical behaviour of our model.

To investigate the singular behaviour of the end-points a and b of the support, or equivalently of the functions s and d given by (2.4), it is convenient to introduce the new variables

$$\sigma \equiv gs \qquad \delta \equiv gd \qquad \tau \equiv \frac{g}{2z}. \tag{3.7}$$

From equation (2.4*a*) we isolate δ

$$\frac{\delta^2}{2} = \frac{\sigma(1+\sigma)(\tau-\sigma) - g^2(L-2)}{1+3\sigma-\tau} \tag{3.8a}$$

which allows us to rewrite (2.4*b*) as a single equation for $\sigma(g^2, L, \tau)$:

$$g^2 L = \frac{\sigma(1+\sigma)(1+2\sigma) - g^2(L-2)}{1+3\sigma-\tau}\sqrt{(\tau-\sigma)^2 - 2\frac{\sigma(1+\sigma)(\tau-\sigma) - g^2(L-2)}{1+3\sigma-\tau}}. \tag{3.8b}$$

A special situation occurs for $1 + 3\sigma - \tau = 0$, and will be discussed later. The above formulae are also the starting point for the expansion in L of the free energy, as discussed in the appendix.

The singular behaviour of the function $\sigma(g^2, L, \tau)$ can be characterized by the condition $\partial(g^2)/\partial\sigma = 0$ which, by equation (3.8*b*), provides a constraint on the parameters. Together, the equations describe in the space (g^2, L, τ), surfaces of criticality $g^2 = g^2_{\mathrm{cr}}(L, \tau)$. Such surfaces will now be investigated in the form of L expansions. We could equally well consider L and g^2 as spectators, and require the condition $\partial\tau/\partial\sigma = 0$, which would provide an identical equation for the critical behaviour. The second equation is

$$\begin{aligned}3g^2L = (6\sigma^2 + 6\sigma + 1)\sqrt{(\tau-\sigma)^2 - \delta^2} + \frac{g^2L(1+3\sigma-\tau)}{(\tau-\sigma)^2-\delta^2}\\ \times\Bigg[\sigma - \tau - \frac{(1+2\sigma)(\tau-\sigma) - \sigma(1+\sigma)}{1+3\sigma-\tau}\\ +3\frac{\sigma(1+\sigma)(\tau-\sigma) + g^2(2-L)}{(1+3\sigma-\tau)^2}\Bigg].\end{aligned} \tag{3.9}$$

In a way fully analogous to the Kazakov model [7] we find three phases for the continuum limit for small values of L.

3.1. The small holes phase

If $\tau > \tau_0 \equiv \frac{1}{2}(\sqrt{3} - 1)$ the set of equations (3.8) and (3.9) allow a Taylor expansion as L approaches zero:

$$\begin{aligned}\sigma(\tau) &= \sigma_0 + L\sigma_1(\tau) + L^2\sigma_2(\tau) + \cdots\\ \delta^2(\tau) &= \delta_0^2 + L\delta_1^2(\tau) + L^2\delta_2^2(\tau) + \cdots\\ g^2_{\mathrm{cr}}(\tau) &= g_0^2 + Lg_1^2(\tau) + L^2g_2^2(\tau) + \cdots\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}&\sigma_0 = \frac{-3+\sqrt{3}}{6} \qquad \delta_0^2 = \frac{1}{3} \qquad g_0^2 = \frac{1}{12\sqrt{3}}\\ &g_1^2(\tau) = \frac{1}{24\sqrt{3}}\left[1 - \sqrt{\frac{2\tau+1-\sqrt{3}}{2\tau+1+1/\sqrt{3}}}\right]\\ &\sigma_1(\tau) = \frac{1}{36}\left[\tau + \frac{\sqrt{3}+1}{2}\right]\left[\tau - \frac{\sqrt{3}-1}{2}\right]^{-1/2}\left[\tau + \frac{\sqrt{3}+3}{6}\right]^{-3/2}.\end{aligned} \tag{3.11}$$

The expansions (3.10) are supposed to hold for $L \leqslant L_{\mathrm{cr}}$. The criticality is very similar to the pure gravity matrix model. From equations (2.8) and (2.9), we find

$$\langle h\rangle \sim \frac{L}{2}\frac{g_1^2(\tau)}{g_0^2} \qquad \langle l\rangle \sim -\frac{L\tau}{2g_0^2}\frac{\partial}{\partial\tau}g_1^2(\tau) \tag{3.12}$$

and the average perimeter of one hole $\langle l\rangle/\langle h\rangle$ approaches a finite constant.

3.2. The tearing phase

If $0 < \tau < \tau_0$, equations (3.8) and (3.9) imply a non-analytic contribution for L close to zero. We find

$$\begin{aligned} \sigma(\tau) &= \sigma_0(\tau) + L^{2/3}\sigma_1(\tau) + \cdots \\ \delta^2(\tau) &= \delta_0^2(\tau) + L^{2/3}\delta_1^2(\tau) + \cdots \\ g_{\mathrm{cr}}^2(\tau) &= g_0^2(\tau) + L^{2/3}g_1^2(\tau) + \cdots \end{aligned} \tag{3.13}$$

with

$$\begin{aligned} \sigma_0(\tau) &= \tfrac{1}{3}\left[\tau - 1 + \sqrt{1 - 2\tau - 2\tau^2}\right] \\ \delta_0^2(\tau) &= -2\sigma_0(\tau)[1 + \sigma_0(\tau)] \\ g_0^2(\tau) &= -\tfrac{1}{2}\sigma_0(\tau)[1 + \sigma_0(\tau)][1 + 2\sigma_0(\tau)]. \end{aligned}$$

The reality of the roots implies the above-mentioned allowed range for τ.

By inserting the expansions, equations (3.13), in the expression of the free energy, equation (2.5), one finds a non-analytic contribution for $E(L)$ at $L = 0$, of the form $E(L) \sim L^{2/3}$:

$$\langle h\rangle \sim L^{2/3} \qquad \langle l\rangle \sim -\frac{\partial \log g_0^2(\tau)}{\partial \log \tau}.$$

The average hole has a diverging perimeter $\langle l\rangle/\langle h\rangle \sim L^{-2/3}$.

The scaling exponents evaluated in the small holes phase and in the tearing phase fully agree with Kazakov results.

3.3. The line separating the two phases

A third phase is found on the line $\tau = 1 + 3\sigma$, sometime referred to as the border phase. The set of equations (3.8) and (3.9) allow a very simple analysis for any value of L. We find

$$\begin{aligned} &54(g^2L)^2\left[1 - \frac{8}{L} + \frac{8}{L^2}\right] + 27(g^2L)^{4/3} - 1 = 0 \\ &\sigma = \tfrac{1}{6}\left[-3 + \sqrt{3 + 18(g^2L)^{2/3}}\right] \\ &\delta^2 = \tfrac{1}{3} + (g^2L)^{2/3} \end{aligned} \tag{3.14}$$

and the following expression for the density, with a non-generic edge behaviour:

$$\rho(\lambda) = \frac{g^2}{2\pi}\frac{(b-\lambda)^{3/2}(\lambda - a)^{3/2}}{1 + 3\sigma - g\lambda}. \tag{3.15}$$

It is also possible to evaluate $\langle l\rangle$ and $\langle h\rangle$ along the whole border line. We calculate the partial derivatives $\partial g^2/\partial\tau$ and $\partial g^2/\partial L$ in equation (3.8a), then we use the criticality condition in the equivalent and more convenient form resulting from the orthogonal polynomial analysis, equation (4.12). For the border line we obtain

$$\begin{aligned} &\langle l\rangle = \frac{\tau}{2g^2}\frac{(g^2L)^{2/3}}{2 - L} \qquad \langle h\rangle = \frac{1}{2}\frac{L}{2 - L} \\ &\frac{\langle l\rangle}{\langle h\rangle} = \tau(g^2L)^{-1/3} = \frac{3\sqrt{2}\tau}{\sqrt{(2\tau + 1 - \sqrt{3})(2\tau + 1 + \sqrt{3})}}. \end{aligned} \tag{3.16}$$

It is remarkable that the value $L = 2$ seems to be the upper value for criticality in this model, as well as in the vector model on random planar graphs [18–20].

For small values of L we find the dominant singular behaviour of the free energy

$$E_{\mathrm{sing}}(L) \sim \text{constant} \times L^{2/3} \qquad \frac{\langle l\rangle}{\langle h\rangle} \sim \tau_0 (g^2 L)^{-1/3}. \tag{3.17}$$

These scaling behaviours differ from those found by Kazakov in phase (iii). In our model the perimeter $\langle l\rangle/\langle h\rangle \sim L^{-1/3}$ exhibits a milder divergence than in Kazakov's model, where it scales as $L^{-2/5}$.

By comparing the expansion for $g^2_{\mathrm{cr}}(\tau)$ in equation (3.10) with the corresponding one, equation (12) in [7], the origin for the different critical exponents in this third phase is apparent: as the point in the parameter space approaches the border line from the critical phase (i), that is $\tau \to \tau_0$, the coefficient $g_1^2(\tau)$ remains finite (although not differentiable), while it diverges as a square root in the Kazakov model. One also sees the existence of a different value $\tau_1 = -\frac{1}{2}(1+\frac{1}{\sqrt{3}})$ which provides the same critical exponents as in Kazakov's model, but is outside the region analysed in this section. We reconsider this point at the end of the next section.

4. The orthogonal polynomial analysis

The qualitative description of the critical behaviour found in the previous section is equivalent to that of Kazakov's model, with the same critical exponents in two critical phases: the small-holes (perturbative) phase and the tearing phase, but a different one in the border phase. This discrepancy deserves a deeper understanding, provided in this section, where the critical behaviour is explored by means of orthogonal polynomials [17, 28]. This independent analysis is more straightforward and it better describes the continuum limit arising from the dynamical triangulation.

After a rescaling $\phi = gM$ the partition function (1.5c) may be written as

$$Z = \int \mathrm{d}\phi\ \exp - \mathrm{tr}\, \frac{N}{g^2}\left(\tfrac{1}{2}\phi^2 + \tfrac{1}{3}\phi^3 + g^2 L \ln(\tau - \phi)\right) = \int \mathrm{d}\phi\ \exp - \mathrm{tr}\, \frac{N}{g^2}\mathcal{V}(\phi). \tag{4.1}$$

Let us introduce the set $\langle\varphi|n\rangle = P_n(\varphi)$ of orthogonal and normalized polynomials

$$\langle m|n\rangle = \int \mathrm{d}\varphi\, \mathrm{e}^{-\mathrm{tr}(N/g^2)\mathcal{V}(\varphi)} P_m(\varphi) P_n(\varphi) = \delta_{mn}. \tag{4.2}$$

The coordinate operator $\hat\varphi : g(\varphi) \to \varphi g(\varphi)$ has the following matrix elements:

$$\langle m|\hat\varphi|n\rangle = \sqrt{R_m}\,\delta_{m,n+1} + S_n \delta_{m,n} + \sqrt{R_n}\,\delta_{m,n-1}. \tag{4.3}$$

The coefficients R_n and S_n are determined by the 'equations of motion':

$$\langle n|\mathcal{V}'(\hat\varphi)|n\rangle = 0 \qquad \langle n-1|\mathcal{V}'(\hat\varphi)|n\rangle = \frac{ng^2}{N\sqrt{R_n}} \tag{4.4}$$

which for our potential have the form

$$0 = S_n + S_n^2 + R_{n+1} + R_n - g^2 L \langle n|\frac{1}{\tau - \hat\varphi}|n\rangle \tag{4.5a}$$

$$\frac{ng^2}{N\sqrt{R_n}} = \sqrt{R_n}(1 + S_n + S_{n-1}) - g^2 L \langle n-1|\frac{1}{\tau - \hat\varphi}|n\rangle. \tag{4.5b}$$

The operator $(\tau - \hat{\varphi})^{-1}$ is the resolvent of a random motion on the lattice $\boldsymbol{N}$. In order to perform the planar limit $N \to \infty$ it is convenient to introduce the conjugate operators $\hat{l}$ and $\hat{\theta}$ [28], defined by

$$\hat{l}|n\rangle = \frac{n}{N}|n\rangle \qquad \mathrm{e}^{\pm\mathrm{i}\hat{\theta}}|n\rangle = |n \pm 1\rangle. \tag{4.6}$$

The operator $\hat{\varphi}$ can be expressed as

$$\hat{\varphi} = \sqrt{R(\hat{l})}\mathrm{e}^{\mathrm{i}\hat{\theta}} + S(\hat{l}) + \mathrm{e}^{-\mathrm{i}\hat{\theta}}\sqrt{R(\hat{l})}. \tag{4.7}$$

In the large-N limit, $\hat{l}$ commutes with $\hat{\theta}$ and can be taken equal to the identity. Assuming the limits $R = \lim_{n\to\infty} R_n$, $S = \lim_{n\to\infty} S_n$, the operator $\hat{\varphi}$ simplifies to

$$\hat{\varphi} = 2\sqrt{R}\cos\theta + S. \tag{4.8}$$

In the planar limit we have

$$\langle n|(\tau - \hat{\varphi})^{-1}|n\rangle \to \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{1}{\tau - \varphi(\theta)} = \frac{1}{\sqrt{(\tau - S)^2 - 4R}} \tag{4.9a}$$

and

$$\langle n-1|(\tau - \hat{\varphi})^{-1}|n\rangle \to \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\tau - \varphi(\theta)} = \frac{1}{2\sqrt{R}}\left(\frac{\tau - S}{\sqrt{(\tau - S)^2 - 4R}} - 1\right). \tag{4.9b}$$

The equations of motion in the planar limit read as

$$0 = S + S^2 + 2R - g^2 L G^{-1}(R, S, \tau) \tag{4.10a}$$

$$g^2 = R + 2RS + \frac{g^2 L}{2}[1 - (\tau - S)G^{-1}(R, S, \tau)] \tag{4.10b}$$

where we denote $G(R, S, \tau) = \sqrt{(\tau - S)^2 - 4R}$. Equations (4.10) correspond to (2.8) with the identifications

$$S = \sigma \qquad R = \frac{\delta^2}{4}$$

and provide $S = S(\tau, g^2, L)$ and $R = R(\tau, g^2, L)$. The continuum limit of the system corresponds to a critical surface in the three-dimensional parameter space spanned by the variables τ, g^2 and L. We can ensure critical behaviour by imposing the following scaling:

$$S = S_0 + S_1 a \qquad R = R_0 + R_1 a \qquad g^2 = g_0^2 + \Lambda a^\ell \qquad g^2 L = \gamma_0 + \Gamma a^k \tag{4.11}$$

with $\ell > 1$ and $k > 1$, where a is a cut-off vanishing in the continuum limit. Indeed (4.11) imply $\partial S/\partial g^2 = \infty = \partial R/\partial g^2$.

The condition $G(R_0, S_0, \tau) \neq 0$ characterizes the perturbative phase. Inserting the scaling laws (4.11) in (4.10) and requiring non-trivial solutions for S_1 and R_1, leads to the equation:

$$4R_0(1 + 3S_0 - \tau)^2 = [(1 + 2S_0)(\tau - S_0) - S_0(1 + S_0) - 6R_0]^2 \tag{4.12}$$

fully equivalent to the critical equation (3.9). As in Kazakov's model [7], the analysis of the critical behaviour is simplified by considering the neighbourhood of $L = 0$. The critical values are $S_0 = \frac{-3+\sqrt{3}}{6}$, $R_0 = \frac{1}{12}$, $g_0^2 = \frac{1}{12\sqrt{3}}$; the consistent value for the exponents ℓ and k is 2. This is the 'small holes' phase.

Let us now consider the non-perturbative phase. When $G(R_0, S_0, \tau)$ tends to zero as $g^2 L$ vanishes a new critical behaviour arises: the phenomenon of spontaneous tearing discussed in [7]. Inserting the scaling laws (4.11) with $\gamma_0 = 0$ and the condition $G(R_0, S_0, \tau) = 0$, in

(4.10) (we observe that in the non-perturbative phase ℓ is not assumed to be greater than one because criticality is ensured by the vanishing of G), we obtain $R_0(\tau)$, $S_0(\tau)$ and $g_0^2(\tau)$ coinciding with $\delta_0^2/4$, σ_0 and g_0^2 given by (3.11), and the equation

$$2\Lambda = \tau(1+3S_0-\tau)[2R_1 - S_1(\tau - S_0)] \tag{4.13}$$

with the constraint $\tau \leqslant \tau_c = (\sqrt{3}-1)/2$ for the non-perturbative phase. The consistent values for ℓ and k are 1 and $\frac{3}{2}$, respectively.

The case $\tau = \tau_c$ (critical tearing) has to be investigated separately, since equation (4.13) implies $\Lambda = 0$ in this limit. Note that (4.10) may be rewritten as

$$R = \frac{1}{2}\frac{S(1+S)(\tau - S) - g^2(L-2)}{1+3S-\tau} \tag{4.14a}$$

$$0 = S(1+S)(1+2S) + g^2(2-L) - g^2L(1+3S-\tau)G^{-1}. \tag{4.14b}$$

If $1+3S_0-\tau_c = 0$ it is straightforward to check that the scaling law compatible with equations (4.14) when $\tau \to \tau_c$ is

$$\begin{aligned} &R = R_0 + R_1 a \qquad S = S_0 + S_1 a \qquad \tau = \tau_c - Ta \\ &g^2 = g_0^2 + \Lambda a^{3/2} \qquad g^2 L = \Gamma a^{3/2} \end{aligned} \tag{4.15}$$

to be compared with the corresponding law in Kazakov's model [8]:

$$\begin{aligned} &R = R_0 + R_1 a \qquad \tau = \tau_c - Ta \\ &g^2 = g_0^2 + \Lambda a^2 \qquad g^2 L = \Gamma a^{5/2}. \end{aligned} \tag{4.16}$$

It follows that in our model the dynamical holes exhibit, in the intermediate phase, a different scaling behaviour with respect to Kazakov's model. Indeed the typical area of the surface diverges at criticality as $1/(g_0^2 - g^2)$, while the total perimeter of the holes on the surface diverges as $1/(\tau_c - \tau)$. Then in the border phase (critical tearing) the scaling laws (4.15) imply for our model that the 'length' of the holes scales as the area to the power of $\frac{2}{3}$, while (4.16) imply for Kazakov's model that in the border phase the length of the holes scales as the square root of the area.

It is interesting to observe that if we defined our model with the potential

$$V_1(M) = \tfrac{1}{2}M^2 - \tfrac{1}{3}gM^3 \qquad g > 0 \tag{4.17}$$

instead of (1.5*b*), the equations corresponding to (4.14) would be

$$R = \frac{1}{2}\frac{S(1-S)(\tau - S) - g^2(L-2)}{1-3S+\tau} \tag{4.18a}$$

$$0 = S(1-S)(1-2S) - g^2(2-L) - g^2L(1-3S+\tau)G^{-1} \tag{4.18b}$$

with the critical values $S_0 = \frac{3-\sqrt{3}}{6}$, $R_0 = \frac{1}{12}$ and $\tau_c = (3+\sqrt{3})/6$ the positive solution of the equation $(\tau_c - S_0)^2 - 4R_0 = 0$. In this case $1 - 3S_0 + \tau_c \neq 0$ and (4.18) admit a scaling law completely analogous to (4.16), implying the same scaling behaviour for the holes as in Kazakov's model even in the border phase.

Let us explain this point. The one-matrix model

$$V(M) = \tfrac{1}{2}M^2 + \tfrac{1}{3}gM^3$$

is invariant under $g \to -g$ and $M \to -M$. Therefore it has two critical points, g^* and $-g^*$, which are equivalent for the purely cubic model and both describe pure gravity. When the random surface is coupled to the holes, the two critical points are no longer equivalent: if the surface reaches the continuum limit by sending g to g^* then holes have the same scaling behaviour as in Kazakov's model, while sending g to $-g^*$, holes have a different scaling behaviour in the border phase. The ‘anomalous’ scaling behaviour of the dynamical holes in the border phase is connected with the following feature of our model when it describes a random surface with a single hole: the absence of the dilute phase for the single static hole interacting with the random surface [29].

5. Asymptotics of orthogonal polynomials

In this section we show the relationship between the L-expansion of the partition function

$$Z_N(L,t) = \int \mathcal{D}M \exp\{-N \operatorname{Tr}[V(M) + L\log(t-M)]\} \tag{5.1}$$

with arbitrary potential $V(M)$, with the asymptotics for large N of the orthogonal polynomial $P_N(\lambda)$ with the measure $\mathrm{e}^{-NV(\lambda)}\,\mathrm{d}\lambda$. Such asymptotic behaviour has recently attracted interest, after the works [30] where it was shown that it provides the (connected) joint probability distributions for the eigenvalues.

The monic polynomial of degree N is given explicitly by the formula [31]

$$P_N(t) = \frac{1}{C}\int \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \prod_{i=1}^{N}(t-\lambda_i) \exp\left[-N\sum_{i=1}^{N} V(\lambda_i)\right] \tag{5.2}$$

where C is the proper normalization factor. The formula can be the starting point for an asymptotics in N, with $t \in (a_0, b_0)$, as investigated by Eynard. We remark that the first-order coefficient $E_1(t)$ provides the *leading* asymptotic behaviour, for large N, of the orthogonal polynomial $P_N(t)$ of the one-matrix model with potential $V(M)$ in the one arc phase.

To take care of the log, the relation is written as follows:

$$P_N(t) = \frac{1}{2C}\left[Z_N\left(-\frac{1}{N}, t+\mathrm{i}\epsilon\right) + Z_N\left(-\frac{1}{N}, t-\mathrm{i}\epsilon\right)\right] \tag{5.3}$$

note that we set $L = -1/N$. For large but finite N, $\log Z_N$ is computed by means of the planar free energy $E_N(-1/N, t) = E_0 - (1/N)E_1(t) + \mathcal{O}(1/N^2)$ where the remainder has the same weight in N as the non-planar terms of the free energy. Using the saddle-point equation for the density $\rho_0(\lambda)$

$$V(t) - 2\int_{a_0}^{b_0} \mathrm{d}\lambda\, \rho_0(\lambda) \log|t-\lambda| = \text{constant}$$

we have, up to irrelevant $2\pi\mathrm{i}$ terms and for t in the support (a_0, b_0) of the density:

$$E_1(t \pm \mathrm{i}\epsilon) = \tfrac{1}{2}V(t) \pm \mathrm{i}\pi \int_{a_0}^{t} \rho_0(s)\,\mathrm{d}s. \tag{5.4}$$

We then find the following leading behaviour, for large N, consistent with the more detailed formula found by Eynard:

$$P_N(t) \approx \exp\left\{\frac{N}{2}(V(t) + c_1)\right\} \cos\left[\pi N \int_{a_0}^{t} \rho_0(s)\,\mathrm{d}s + \mathcal{O}(1)\right] \tag{5.5}$$

where c_1 is a constant which depends on the normalization for P_N and, in particular, it vanishes for P_N not monic but with unit norm. The omitted terms $\mathcal{O}(1)$ cannot be accounted for by a planar calculation, since they would require the contribution from the graphs on the torus and higher genera.

6. Conclusions

In this paper we analysed a model of random surfaces with coherently oriented boundaries, defined in (1.5), evaluated the free energy in the spherical limit, equation (2.5), and the continuum limits associated to the phases of the model. Both the model and its analysis are parallel to Kazakov's model [7] where the boundaries are not oriented. The phase diagram for the two models are similar. Figure 3 shows the phase diagram of Kazakov's model. Critical behaviour occurs only below the line $L = 4/z^2$, with three inequivalent continuum limits. The phase describing surfaces with small holes is for $g/z^2 > 2/3$, whereas the 'tearing phase' is for $g/z^2 < 2/3$. The boundary line is the vertical segment $g/z^2 = 2/3$.

Figure 4 is the phase diagram of the model analysed in this paper. The plotted line, with equation

$$\tau = 1 + 3\sigma = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3 + 18(g^2 L)^{2/3}} \tag{6.1}$$

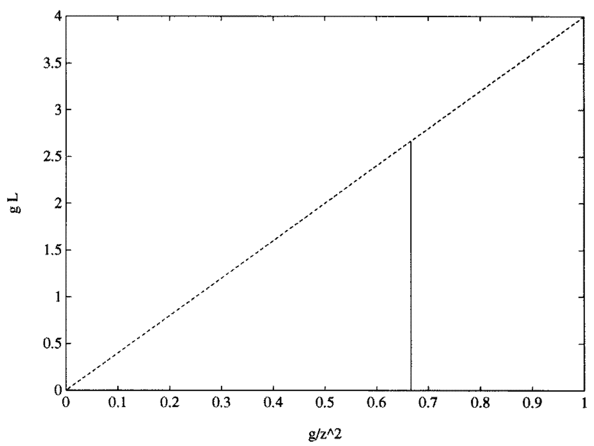

Figure 3. Phase diagram of Kazakov's model. The continuous line has equation $g/z^2 = 2/3$.

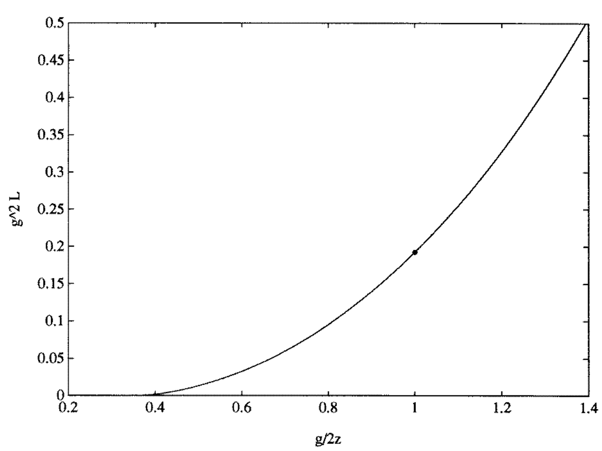

Figure 4. Phase diagram of the present model. On the horizontal axis are the values of $\tau = g/(2z)$. The curve originates at $\tau = (\sqrt{3} - 1)/2$, corresponding to the critical values of the parameters of the one-matrix model with cubic interaction.

is the boundary between a 'small hole phase' (right) and a 'tearing phase' (left). In both the above phases we find that the total perimeter of the holes $\langle l\rangle$, as well as the average number of holes $\langle h\rangle$, have the same critical exponents in terms of the fugacity L that appear in the model [7], thus confirming the universality of Kazakov's analysis.

However, in sections 3 and 4 it is carefully shown that the boundary phase (6.1) has critical exponents and scaling behaviour inequivalent to the corresponding ones of the boundary phase of Kazakov's model. This difference is due to partial cancellations in the perturbative expansion of the free energy. The model with the opposite sign of the cubic coupling, equation (4.17), does not have such cancellations and it has the same critical exponents of Kazakov's model in all three phases.

The model analysed in this paper is also related to the $O(n)$ vector model on random surfaces [18–20] which, in the spherical limit, describes non-intersecting loops drawn on trivalent planar graphs. Those loops are not oriented and may include part of the trivalent graph and/or other loops, unlike the coherently oriented loops of the present model, which are microscopic or macroscopic holes. The complete analytic solution of the $O(n)$ model is not yet available in the spherical limit and, because of the relation mentioned at the end of the introduction, the analytic solution of the present, much simpler model, may be a step towards it.

Acknowledgments

One of us, GMC, would like to thank Professor R Blankenbecler for the kind hospitality at SLAC, where part of this work was done and Dr A Zee for a useful conversation about [30].

Appendix

We here assume that the free energy, equation (2.5), and the parameters $\sigma = gs$, $\delta = gd$ computed by (3.8), allow a formal Taylor expansion around $L = 0$, and we quote the first term E_1. Setting

$$\sigma(g, L, \tau) = \sigma_0(g) + L\sigma_1(g, \tau) + L^2\sigma_2(g, \tau) + \cdots \tag{A.1a}$$

$$\delta(g, L, \tau) = \delta_0(g) + L\delta_1(g, \tau) + L^2\delta_2(g, \tau) + \cdots \tag{A.1b}$$

we find, at the lowest order, the parameters of the pure cubic model: $\sigma_0(g)$ solves the cubic equation

$$\sigma_0(1 + \sigma_0)(1 + 2\sigma_0) + 2g^2 = 0 \tag{A.2a}$$

$$\delta_0^2 = -2\sigma_0(1 + \sigma_0). \tag{A.2b}$$

At the next order we give σ_1 only:

$$\sigma_1(g, \tau) = \frac{g^2}{1 + 6\sigma_0 + 6\sigma_0^2}\left[1 + \frac{(1 - \tau + 3\sigma_0)}{\sqrt{(\tau - \sigma_0)^2 + 2\sigma_0(1 + \sigma_0)}}\right]. \tag{A.3}$$

The above coefficients lead to the evaluation of the first two coefficients of the L expansion of the free energy

$$E_0(g) = \frac{1}{2}\log[1 + 2\sigma_0] - \frac{1}{3}\sigma_0\frac{2 + 6\sigma_0 + 3\sigma_0^2}{(1 + 2\sigma_0)^2(1 + \sigma_0)} \tag{A.4}$$

which obviously reproduces the well known free energy of the cubic model [17], and

$$E_1(g,\tau) = -2\log(2g) + \log\left[(\tau-\sigma_0) + \sqrt{(\tau-\sigma_0)^2-\delta_0^2}\right] - 1 - \frac{1}{6g^2}\sigma_0^2(3+2\sigma_0)$$
$$+\frac{1}{g^2}\left(\frac{1}{2}\tau^2+\frac{1}{3}\tau^3\right) + \left[\frac{1}{3}\frac{\sigma_0(2+3\sigma_0)+\tau(6\sigma_0^2+11\sigma_0+6)}{(1+2\sigma_0)(1+\sigma_0)}\right.$$
$$\left.+\frac{1}{6g^2}(\tau^2\sigma_0(1-2\sigma_0)+\tau^3(2\sigma_0-3)-2\tau^4)\right]\frac{1}{\sqrt{(\tau-\sigma_0)^2-\delta_0^2}}. \qquad \text{(A.5)}$$

References

[1] During the couple of years that led to the present work, we circulated two preprints hep-th 9412080 and hep-th 9505137. We think that the present edition, which benefitted from the criticism, suggestions and encouragements of some referees, is a more coherent and complete presentation of our work.
[2] Wadja S 1981 *Phys. Rev.* D **24** 970
[3] Migdal A 1983 *Phys. Rep.* **102** 199
[4] David F 1990 *Mod. Phys. Lett.* A **5** 1019
[5] Ambjorn J *et al* 1990 *Mod. Phys. Lett.* A **5** 1753; *Phys. Lett.* **251B** 517
[6] Duplantier B and Kostov I 1990 *Nucl. Phys.* B **340** 491
[7] Kazakov V A 1990 *Phys. Lett.* **237B** 212
[8] Kostov I K 1990 *Phys. Lett.* **238B** 181
[9] Minahan J A 1991 *Phys. Lett.* **268B** 29
[10] Yang Z 1991 *Phys. Lett.* **257B** 40
[11] Minahan J A 1993 *Int. J. Mod. Phys.* A **8** 3599
[12] Minahan J A 1991 *Phys. Lett.* **265B** 382; 1992 *Nucl. Phys.* B **378** 501
[13] Rusakov B 1990 *Mod. Phys. Lett.* A **5** 693
[14] Gross D 1993 *Nucl. Phys.* B **400** 161
[15] Gross D and Taylor W IV 1993 *Nucl. Phys.* B **400** 181; **403** 395
[16] Caselle M, D'Adda A, Magnea L and Panzeri S 1994 *Nucl. Phys.* B **416** 751
[17] Brezin E, Itzykson C, Parisi G and Zuber Z B 1978 *Commun. Math. Phys.* **59** 35
Bessis D, Itzykson C and Zuber J B 1980 *Adv. Appl. Math.* **1** 109
[18] Kostov I K 1989 *Mod. Phys. Lett.* A **4** 217
[19] Gaudin M and Kostov I 1989 *Phys. Lett.* **220B** 200
[20] Kostov I and Staudacher M 1992 *Nucl. Phys.* B **384** 459
[21] Barbieri A, Cicuta G M and Montaldi E 1984 *Nuovo Cimento* A **84** 173
[22] Cicuta G M, Molinari L, Montaldi E and Riva F 1987 *J. Math. Phys.* **28** 1716
[23] Anderson A, Myers R C and Periwal V 1991 *Phys. Lett.* **254B** 89; *Nucl. Phys.* B **360** 463
Myers R C and Periwal V 1993 *Nucl. Phys.* B **390** 716
[24] Itzykson C and Zuber J B 1980 *J. Math. Phys.* **21** 411
[25] Muskhelishvili N I 1953 *Singular Integral Equations* ed P Noordhoff
[26] Das S R, Dhar A, Sengupta A N and Wadia S R 1990 *Mod. Phys. Lett.* A **5** 1041
[27] Cicuta G M and Montaldi E 1990 *Mod. Phys. Lett.* A **5** 1927
[28] Alvarez O and Windey P 1991 *Nucl. Phys.* B **348** 490
[29] Cicuta G M, Molinari L and Stramaglia S 1995 Universality of the tearing phase in matrix models *Preprint* hep-th 9505137
[30] Brezin E and Zee A 1993 *Nucl. Phys.* B **402** 613; 1994 *Nucl. Phys.* B **424** 435
Eynard B 1993 *Saclay Preprint* SphT/93-999
[31] Szego G 1939 *Orthogonal Polynomials* (American Math. Soc. Coll. Pub. **23**)